QUANTIFYING RISK

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I. Introduction

II. Requirement of the Basel agreements

III. Coherent Risk measures
INTRODUCTION AND OVERVIEW

- Historical remarks
- Institutional framework
- Basel agreements
THE PROBLEM

• Consider:
  – Investor who owns the stock of a firm
  – Insurance company who sells an insurance policy
  – Private person who wants to change a fixed IR mortgage into one with variable IR

• Important common aspect:
  One owns today a security whose future value is uncertain
MODELING RISK

- Example:
  - Stock: price risk
  - Bank credit: Interest rate risk, default risk
  - Insurance policy: risk that the insurance company must pay.

- Change/hazard is an important element in the valuation problem.

- Risk in its most general form can be modeled by a random variable
MODELING RISK (CONT.)

- Concerns in most cases the distribution function
  \[ F_X(x) = P(X \leq x) \].

- Special risk measures: Statistics defined on \( F_X \)
  (i.e., functions of the random variable \( X \))

- Examples:
  - Mean: \( \mu_X = E[X] \)
  - Standard deviation \( \sigma_X = \sqrt{\text{var}(X)} \)
  - Quantile: \( q_\alpha, \alpha \in [0, 1]: P(X > q_\alpha) = 1 - \alpha \).
  - Value at Risk (VaR): VaR
  - Expected Shortfall.

All these measures are important
MEANING OF RISK

• The Meaning of risk depends on the point of view.

• Examples:
The trading book of a commercial bank.

• Internal view of the bank:
  – Maximisation of the present value
  – People “know” that there is no free lunch
  – Time aspect:
    Different instruments have different holding periods.
    The important quantities:
    * Future value
    * Future risk
MEANING OF RISK (CONT.)

- Other parts concerned:
  - Share holders have other interests (maximisation of stock prices or dividends)
  - The public (represented through the regulatory bodies) would like to prevent that the economy is affected in case of bankruptcy of the bank

- Modern risk management takes place in coordination of different groups of interest of the society to which the bank belongs

- Important not only qualitative rules (i.e., laws) but more and more quantitative methods of best practice
MEANING OF RISK (CONT.)

- Important question for the bank:
  How can the risk manager describe and control the risk-profit profile in a way that the interests of all parts concerned (including the regulators) are satisfied?

- Complex optimisation problem

- Important aspects:
  - Quantification of risk
  - Protection against risk or risk transfer with financial instruments
HISTORICAL COMMENTS

• Risk management exists since a long time

• Example of an option 1800 v. Chr. (book of Hammurabi:)
  If anyone owe a debt for a loan, or the harvest fail, or the
  grain does not grow for lack of water; in that year he need
  not give his creditor any grain, he washes his debt-tablet
  in water and pays no rent for this year.
  I.e. option of the debtor not to pay interest if the harvest fails.

• Today there are
  – Weather options
  – WINCAT (Winterthur Catastrophy Options)

• In Amsterdam, 17th century, there where Call- and put options
HISTORICAL COMMENTS (CONT.)

• BUT:
  Description of risk through random variables and quantifying risk through statistics of the distribution function is in historical perspective not trivial.

• Systematische development of probability theory only since the 17th century

• Systematic development of Statistics since the 19th century

• Mathematical modeling in Finance and insurance: 20th century
  (mostly developed during the last 20 to 30 years)
HISTORICAL COMMENTS (CONT.)

- before 1960: Cash, stocks, loans, mortgages, bonds

- after 1970: A whole “zoo” of instruments has been created:
  - Futures, Optionen (unterschiedlichster Ausführung: European, American, Asian, Barrier, Digital, etc.)
  - Swaps, Swaptions, etc. (Zinsderivate)
  - Asset-backed securities, mortgage-backed Securities
  - Kreditderivate
EXAMPLE EUROPEAN CALL OPTION

- The “drosophila” of financial mathematics

- Price of a Call option:

\[ C(T) = \max(S_T - K, 0) = (S_T - K)^+ \]

with

- \( T \) Maturity
- \( K \) Strike price
- \( S_T \) Stock price at maturity \( T \).
EXAMPLE EUROPEAN CALL OPTION (CONT.)

- Solution is the well-known Black & Scholes formula that has revolutionized the world (of finance):

\[ C(S, t) = SN(d_1) - Ke^{-r(T-t)}N(d_2) \]

where

\[ d_1 = \frac{\log(S/K) + (r + \sigma^2/2)(T-t)}{\sigma \sqrt{T-t}} \]

\[ d_2 = \frac{\log(S/K) + (r - \sigma^2/2)(T-t)}{\sigma \sqrt{T-t}} = d_1 - \sigma \sqrt{T-t} \]

and \( N(x) \) is the standardised normal distribution.

- Options are considered as one of the key innovations of the 20th century
1973 Chicapo Option Exchange opened. Volume: < 1000 options/day

1995 Volume increased to > 1 000 000 options/day
For comparison: the power of computers increased even faster (Moore’s law).

- In view of this fast development the occurrence of accidents is not surprising.
Nur schlaglichtartig, unvollständig.

- Crises:
  1987 Stock market crash, triggered by automatic trading programs (portfolio insurance)
  1996 Barings bank
  1998 LTCM
  2001–03 “Smooth crash” of the world stock markets Speculative bankruptcies (ENRON, Worldcom, Swissair).
MEASURES OF THE REGULATORS

1974 Foundation of the Basel Committee of Banking Supervision
(1 year after floating of the FX rates).
Cannot make laws but state

   broad standards and rules and advice according to
   best practice.

1988 Basel I.
First step to a minimal international standard risk capital of
bank
Insufficient from hindsight
1993 G-30 report for the first time systematically deals with “off-balance” products as derivatives

J.P. Morgan introduces a daily 1-page risk report and introduces the Value-at-Risk as risk measure

J.P. Morgan develops RiskMetrics as model to determine marked risk

Concepts as ”‘mark-to-market”’ (instead of book values) are accepted to be important

The regulators must act.
MEASURES OF THE REGULATORS (CONT.)

1996 Important amendment of the Basel I agreement:
Quantifying market risk with a standard VaR model with the possibility to use an internal market VaR model → stimulated technological development of quantifying market risks.

1999 First consultative paper on revising the Capital Accord (Basel II)

2004 Publication of the new Capital Accord.

2007 Entry into force of Basel II.

- Basel agreement concerns mainly banks.
- In the insurance sector: Solvency II (EU) or Swiss Solvency Test (SST).
TYPES OF RISK

For financial institution:

• Market risk
due to changes in market conditions (FX, IR, equities, commodities)

• Credit risk
due to the fact that a debtor cannot fulfil his obligations

• Operational Risk
  (example: IT, error of employees, etc.)

• Liquidity risk

• Model risk

• Parameter risk
Consequences of Basel ’96

“Quantum leap” in the importance of quantitative risk modeling in financial institutions:

- CEO is responsible
- Generally accepted guidelines and
- Position of CRO (Chief Risk Officers) has been created
- In the beginning mainly market risk was considered liquidity risk is related (→ black Monday)
Consequences of Basel ’96 (CONT.)

BUT: The main result of modern risk management is that a wide spectrum of derivate products has been created with exactly the aim to bring liquidity to the market through instruments tailored towards investments, hedging and risk transfer needs.

- Greenspan, speech 02: Aim has mainly been reached. (there has been only a relatively small reaction of the real economy to the ”‘smooth crash”’ of the stock markets and the bankruptcies during 2001–2003.

- However:
  - For credit risk we are still in this process
  - For operational risk, we are at the beginning
Basel 96 Agreement

Two approaches to compute risk capital

1. Standard model
2. Internal Model
Internal Model

4 risk categories:

- Interest rates
- Equities
- FX
- Commodities
Internal Model

- 99%-VaR for 10 days
  (measure for risk not to liquidate a position within 10 days)

\[
C_t = \max \left\{ \text{VaR}_{t-1} + d_t \text{ASR}^{\text{VaR}}_{t-1} \right. \\
\left. + \frac{1}{60} \left[ m_t \sum_{j=1}^{60} \text{VaR}_{t-j} + d_t \sum_{j=1}^{60} \text{ASR}^{\text{VaR}}_{t-j} \right] \right\}
\]

with

\text{VaR}_{t-i} \ 99\%, \ 10\text{-}Tage \text{ VaR} \ at \ day \ t - i,

\[m_t \ \text{multiplicator for day} \ t, \ m_t \geq 3 \ (\text{depends on the statistical quality of the model})\]

\[\text{ASR}^{\text{VaR}} \ \text{Extra VaR-based charge for special risks}\]

\[d_t \in \{0, 1\} \ \text{Indikator function indicating at which days there has been special risks}\]
<table>
<thead>
<tr>
<th>Number of Exceptions</th>
<th>Multiplier</th>
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<tbody>
<tr>
<td>4 or fewer</td>
<td>3.00</td>
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<tr>
<td>5</td>
<td>3.40</td>
</tr>
<tr>
<td>6</td>
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<td>7</td>
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<td>8</td>
<td>3.75</td>
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<tr>
<td>9</td>
<td>3.85</td>
</tr>
<tr>
<td>10 or more</td>
<td>4.00</td>
</tr>
</tbody>
</table>

Multiplier Based on the Number of Exceptions in Back-Testing
NEW REQUIREMENTS FOR CREDIT RISK

- Capital requirement for credit risk depends on the rating of the counter part
- 99.9 Value of Risk with a time horizon of 1 Year has to be calculated.
RISK MEASURES

• The Problem:
  How to measure the risk of a (portfolio of) risky asset(s)?

• Stock index (reflects the value of a “standard” portfolio)
### Statistical Properties of Daily and Weekly DAX Returns

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<thead>
<tr>
<th></th>
<th>täglich</th>
<th>wöchentlich</th>
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<tbody>
<tr>
<td>Min.</td>
<td>-0.0995</td>
<td>-0.1718</td>
</tr>
<tr>
<td>1st Qu.</td>
<td>-0.0034</td>
<td>-0.0137</td>
</tr>
<tr>
<td>Median</td>
<td>0.0001</td>
<td>0.0037</td>
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<tr>
<td>Mean</td>
<td>0.0002</td>
<td>0.0022</td>
</tr>
<tr>
<td>3rd Qu.</td>
<td>0.0043</td>
<td>0.0204</td>
</tr>
<tr>
<td>Max.</td>
<td>0.0685</td>
<td>0.1261</td>
</tr>
<tr>
<td>Stdev.</td>
<td>0.0092</td>
<td>0.0320</td>
</tr>
</tbody>
</table>
CDAX: Verteilungsfunktion für tägliche DAX-Renditen
CDAX: Verteilungsfunktion für 2-wöchentliche DAX-Renditen
RISK MEASURES (CONT.)

- Standard deviation not adequate because possible skewness and heavy tails
- In risk management: Focus on left tail of return distribution
- Question:
  How much capital is needed in order not to have bad surprises?

\[
\sup\{r \in R : F_R(r) = 0\}
\]

This is in principle infinity (not realistic).
• According to Basel agreements: Risk capital should be larger than the largest loss that is not exceeded with a certain probability (confidence level).
• This is the qualitative meaning of Value at Risk.
Mathematical Definition of Value at Risk

• Given
  – a portfolio of risky assets
  – a returns distribution $F_R(r) = P(R \leq r)$ for a fixed time horizon $\Delta t$
  – the confidence level $\alpha$.

• The Value of Risk with confidence level $\alpha$ (VaR$_\alpha$) is defined as

$$\text{VaR}_\alpha(R_{t,\Delta t}) = -\inf\{r \in \mathbb{R} : P(R_{t,\Delta t} \leq r) \geq 1 - \alpha\}$$

$$= -\sup\{r \in \mathbb{R} : P(R_{t,\Delta t} \leq r) < 1 - \alpha\}$$
PROBABILISTIC INTERPRETATION OF VaR

From a probabilistic point of view, VaR is the negative of the $(1 - \alpha)$-quantile of the generalized distribution function of the log. returns:

$$q_{1-\alpha}(F) := F^{-1}(1 - \alpha) = \inf\{r \in R : F(r) \geq 1 - \alpha\},$$

Therefore:

$$\text{VaR}_\alpha = -\inf\{r \in R : F_R(r) \geq 1 - \alpha\} = -q_{1-\alpha}(F_R)$$
Computation of the VaR

- For normal distribution VaR can be computed by means of the error function
  \[
  \text{VaR}_\alpha = -(\mu_L + \sigma_L q_\alpha(\Phi))
  \]
  with the error function
  \[
  \Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt.
  \]

- Returns are in general not normally distributed
  - Market risk: heavy tails
  - Credit Risk: very skewed distributions
VAR: COMMENTS

- In general VaR cannot be computed analytically.
  - Numerical algorithms can be used if the distribution function is known.
  - Stochastic simulations (Monte Carlo methods) are needed if the distribution is not known.
- To determine the risk capital, the VaR is often computed with respect to the mean value of the expected return:
  \[
  \text{VaR}_{\alpha}^{\text{Mean}} := \text{VaR}_{\alpha} - \mu_L. \tag{1}
  \]
  The difference is important, if \( \Delta t \) and consequently \( \mu_L[\Delta t] \) is large, as e.g. for credit risk (\( \Delta t = \text{Jahr} \)).
CHOICE OF THE PARAMETER $\Delta t$

- Depends on the time span, after which the portfolio is reallocated.
  - Market risk of banks: $\Delta t = 1 \ldots 10$ days
  - Credit risk: $\Delta t = 1 \ldots 1$ year
  - Insurance companies: $\Delta t = 1 \ldots 1$ year
- For technical reasons VaR can be easier computed for small $\Delta t$. E.g., estimations are more reliable because of the larger sample size.
CHOICE OF THE PARAMETER $\alpha$

- If VaR is used to for model calibration or to limit positions, $\alpha = 95\%$ is a commonly chosen value (1 event out of 20)

- If VaR is computed to determine the risk capital $\alpha$ has to be large enough.
  - Market risk: $\alpha = 99\%$
  - Credit risk: $\alpha = 99.9\%$

- It’s important to keep in mind the limit of the VaR concept: sample size has to be large enough
DISGRESSION MARKET RISK: CHARACTERIZATION OF THE TAILS

- Classification of distributions
- Characterisation of fat-tailed distributions
- Empirical estimation of tail behavior
- Importance of the sample size
- How heavy are the tails of financial instruments?
- Aggregation properties
CLASSIFICATION OF DISTRIBUTIONS

- Thin-tailed distributions:
  - Cumulative distribution function declines exponentially in the tails
  - All moments are finite

- Fat-tailed distributions:
  - Cumulative distribution function asymptotically declines as a power with exponent $\alpha$

- Bounded distributions without tails
MOMENTS OF FAT-TAILED DISTRIBUTIONS

- The density of fat-tailed distributions asymptotically behave as
  \[ \phi(x) = a (x - x_0)^{-\alpha - 1} + o((x - x_0)^{-\alpha - 1}) \]

- Moments:
  \[ E[X^k] = C + a \int_y^\infty (x - x_0)^{-\alpha + k - 1} \, dx \]
  \[ + \, o((x - x_0)^{k - \alpha}) \]
  only exist for \( k < \alpha \).

- Tail index stable under aggregation.
HOW HEAVY ARE TAILS OF FINANCIAL DISTRIBUTIONS?

- Mandelbrot: Returns follow stable distributions.
  Implications:
  - Tail indices between 0 and 2
  - Variance does not exist.

- Risk has often be associated with variance
  (e.g. Markowitz portfolio theory)

- Existence of variance central for all areas of finance

- Try to decide this question empirically
EMPIRICAL ESTIMATION OF TAIL BEHAVIOR

• Let $X_1, X_2, \ldots, X_n$ a sample of $n$ independent observations with unknown probability distribution function $F$.

• Let $X_{(1)} \geq X_{(1)} \geq \ldots \geq X_{(n)}$ the descending order statistics.

• The so-called Hill-estimator (Hill, 1975)

$$\hat{\gamma}^{H}_{n,m} = \frac{1}{m - 1} \sum_{i=1}^{m-1} \ln X_{(i)} - \ln X_{(m)}$$

with $m > 1$ is a consistent estimator of $\gamma = 1/\alpha$. 

Dynamics of Socio-Economic Systems... DPG — Schools on Physics 2005
Sample size: 100000, average computed over 50 simulations.
HILL-ESTIMATOR: PROPERTIES

• \((\gamma_{n,m} - \gamma) \sqrt{m}\) is asymptotically normally distributed.
• \(\hat{\gamma}_{n,m}^H\) biased for finite sample size.
• \(\hat{\gamma}_{n,m}^H\) depends on \(m\).
• No easy way to determine best value of \(m\)
EMPIRICAL RESULTS

• Estimated by a subsample bootstrap method

• Confidence ranges are standard errors times 1.96
  (would correspond to 95% confidence for normally-distributed errors)

• Computation of standard error by jackknife method:
  – Data sample modified in 10 different ways by removing one-tenth of the total sample
  – Tail index computed separately for every modified sample
### TAIL INDICES FOR MAIN FX RATES

<table>
<thead>
<tr>
<th>Rate</th>
<th>30m</th>
<th>1 hr</th>
<th>2 hr</th>
<th>6h</th>
<th>1 day</th>
</tr>
</thead>
<tbody>
<tr>
<td>USD_DEM</td>
<td>3.18 ± 0.42</td>
<td>3.24 ± 0.57</td>
<td>3.57 ± 0.90</td>
<td>4.19 ± 1.82</td>
<td>5.70 ± 4.39</td>
</tr>
<tr>
<td>USD_JPY</td>
<td>3.19 ± 0.48</td>
<td>3.65 ± 0.79</td>
<td>3.80 ± 1.08</td>
<td>4.40 ± 2.13</td>
<td>4.42 ± 2.98</td>
</tr>
<tr>
<td>GBP_USD</td>
<td>3.58 ± 0.53</td>
<td>3.55 ± 0.65</td>
<td>3.72 ± 1.00</td>
<td>4.58 ± 2.34</td>
<td>5.23 ± 3.77</td>
</tr>
<tr>
<td>USD_CHF</td>
<td>3.46 ± 0.49</td>
<td>3.67 ± 0.77</td>
<td>3.70 ± 1.09</td>
<td>4.13 ± 1.77</td>
<td>5.65 ± 4.21</td>
</tr>
<tr>
<td>USD_FRF</td>
<td>3.43 ± 0.52</td>
<td>3.67 ± 0.84</td>
<td>3.54 ± 0.97</td>
<td>4.27 ± 1.94</td>
<td>5.60 ± 4.25</td>
</tr>
<tr>
<td>USD_ITL</td>
<td>3.36 ± 0.45</td>
<td>3.08 ± 0.49</td>
<td>3.27 ± 0.79</td>
<td>3.57 ± 1.35</td>
<td>4.18 ± 2.44</td>
</tr>
<tr>
<td>USD_NLG</td>
<td>3.55 ± 0.57</td>
<td>3.43 ± 0.62</td>
<td>3.36 ± 0.92</td>
<td>4.34 ± 1.95</td>
<td>6.29 ± 4.96</td>
</tr>
<tr>
<td>XAU_USD</td>
<td>4.47 ± 1.15</td>
<td>3.96 ± 1.13</td>
<td>4.36 ± 1.82</td>
<td>4.13 ± 2.22</td>
<td>4.40 ± 2.98</td>
</tr>
<tr>
<td>XAG_USD</td>
<td>5.37 ± 1.55</td>
<td>4.73 ± 1.93</td>
<td>3.70 ± 1.52</td>
<td>3.45 ± 1.35</td>
<td>3.46 ± 1.97</td>
</tr>
</tbody>
</table>
# TAIL INDICES FOR SOME CROSS RATES

<table>
<thead>
<tr>
<th>Rate</th>
<th>30m</th>
<th>1 hr</th>
<th>2 hr</th>
<th>6h</th>
</tr>
</thead>
<tbody>
<tr>
<td>DEM_JPY</td>
<td>4.17 ± 1.13</td>
<td>4.22 ± 1.48</td>
<td>5.06 ± 1.40</td>
<td>4.73 ± 2.19</td>
</tr>
<tr>
<td>GBPDEM</td>
<td>3.63 ± 0.46</td>
<td>4.09 ± 1.98</td>
<td>4.78 ± 1.60</td>
<td>3.22 ± 0.72</td>
</tr>
<tr>
<td>GBP_JPY</td>
<td>3.93 ± 1.16</td>
<td>4.48 ± 1.20</td>
<td>4.67 ± 1.94</td>
<td>5.60 ± 2.56</td>
</tr>
<tr>
<td>DEM_CHF</td>
<td>3.76 ± 0.79</td>
<td>3.64 ± 0.71</td>
<td>4.02 ± 1.52</td>
<td>6.02 ± 2.91</td>
</tr>
<tr>
<td>GBP_FRF</td>
<td>3.30 ± 0.41</td>
<td>3.42 ± 0.97</td>
<td>3.80 ± 1.34</td>
<td>3.48 ± 1.75</td>
</tr>
<tr>
<td>FRFDEM</td>
<td>2.56 ± 0.34</td>
<td>2.41 ± 0.14</td>
<td>2.36 ± 0.27</td>
<td>3.66 ± 1.17</td>
</tr>
<tr>
<td>DEM_ITL</td>
<td>2.93 ± 1.01</td>
<td>2.60 ± 0.66</td>
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<td>2.76 ± 1.49</td>
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<td>DEM_NLG</td>
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<td>2.19 ± 0.13</td>
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<td>3.24 ± 0.87</td>
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<tr>
<td>FRF_ITL</td>
<td>2.89 ± 0.34</td>
<td>2.73 ± 0.49</td>
<td>2.56 ± 0.41</td>
<td>2.34 ± 0.66</td>
</tr>
</tbody>
</table>
RESULTS: SUMMARY

• For 30 min returns all rates against USD as well as nearly all cross rates outside the EMS have tail indices around 3.5 (between 3.1 and 3.9).

• Gold and silver have tail indices above 4. These markets differ from the FX market, with very high volatilities in the 80s and much lower volatilities in the 90s (structural change).

• EMS rates show lower tail indices around 2.7:
  → Reduced volatility induced by the EMS rules is at the cost of a larger probability of extreme events for realigning the system.
  → Argument against the credibility of systems like the EMS.
• Tail index reflects:
  – Institutional setup
  – The way different agents interact
• They are an empirical measure of market regulation and efficiency
• Tail index large:
  – Free interactions between agents with different time horizons
  – Low degree of regulation
  – Smooth adjustment to external shocks
• Tail index small:
  – High degree of regulation
  – Abrupt adjustment to external shocks
• Tail indices between 2 and 4
• 2nd moments of return distributions finite
• 4th moments of return distributions usually diverge
• → Absolute returns are preferred for computation of volatility autocorrelation.
RESULTS: SUMMARY (CONT.)

- Return distributions are fat-tailed with tail index $\geq 2$ and therefore non-stable.

- Invariance under aggregation satisfied up to $2 \ldots 6$ hours.

- For larger time horizons tail behavior can no longer be observed because:
  - Transition to tail behavior occurs at increasingly larger (relative) returns.
  - Sample size decreases.

- Detailed studies have shown that 18 years of daily data are not enough to estimated the tail index reliably.
Density of $t$-3.5 distribution (black squares), sum of 4 $t$-3.5 distributed random variables (+) and sum of 10 $t$-3.5 distributed random variables. Note the transition from normal to tail behavior.
EXTREME RISKS IN FINANCIAL MARKETS

- What are the extreme movements to be expected?
- Are there theoretical processes to model them?
- What is the best hedging strategy?
## EXTREME EVENTS IN MODELS AND REALITY

<table>
<thead>
<tr>
<th>Probabilities [1/years]</th>
<th>1/1</th>
<th>1/5</th>
<th>1/10</th>
<th>1/15</th>
<th>1/20</th>
<th>1/25</th>
</tr>
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<tbody>
<tr>
<td>Normal</td>
<td>0.4%</td>
<td>0.5%</td>
<td>0.6%</td>
<td>0.6%</td>
<td>0.7%</td>
<td>0.7%</td>
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<tr>
<td>Student-3</td>
<td>0.5%</td>
<td>0.8%</td>
<td>1.0%</td>
<td>1.1%</td>
<td>1.2%</td>
<td>1.2%</td>
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<tr>
<td>GARCH(1,1)</td>
<td>1.5%</td>
<td>2.1%</td>
<td>2.4%</td>
<td>2.6%</td>
<td>2.7%</td>
<td>2.9%</td>
</tr>
<tr>
<td>USD_DEM</td>
<td>1.7%</td>
<td>2.5%</td>
<td>3.0%</td>
<td>3.3%</td>
<td>3.5%</td>
<td>3.7%</td>
</tr>
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<td>USD_JPY</td>
<td>1.7%</td>
<td>2.4%</td>
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</tr>
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<td>GBP_USD</td>
<td>1.6%</td>
<td>2.3%</td>
<td>2.6%</td>
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LIMITS OF THE VAR

- A caveat
- Assumptions
- Sources of errors
**LIMITS OF THE VAR: A CAVEAT**

- Often VaR is often interpreted as:
  
  With probability $\alpha$ the expected loss is not greater as $VaR_{\alpha}$.

- However, this interpretation is **dangerous**:
  
  Various sources of error can make this literal interpretation **wrong**
LIMITS OF THE VAR: ASSUMPTIONS

- Sufficiently large sample size
- Stationarity (can be a problem for a time series of 30 years)
- Independence of the (extreme) events
LIMITS OF THE VAR: POSSIBLE SOURCES OF ERRORS

- **Estimation risk, model risk** (i.e., the form of the loss distribution and the reliability of the parameter estimation)

- **Market liquidity:** If one is forced to close a position there are *additional* liquidity costs. This is important in particular for big positions
  (*Example:* LTCM)

- \( \text{VaR}_\alpha \) contains no information about the typical size of a loss given that it exceeds \( \text{VaR}_\alpha \).

- In certain cases: Definition is **inadequate**
EXAMPLE:
VAR FOR A PORTFOLIO OF SPECULATIVE BONDS

Assumptions:
• Given 50 different bonds with face value 100 which is returned at maturity \( T = t + \Delta t \) (e.g. \( \Delta t = 1 \) year).

• The Coupon payment (interest rate) is 5%.

• All bonds have the same default probability of 2%.

• Defaults of several bonds are pairwise independent.

• If a bond defaults there is no recovery (i.e., the whole face value is lost).
EXAMPLE (CONT.)

- Market value of the bond at $t$ be 95. 
  (Corresponds to time to maturity of one year and 5% market IR)

- $\forall i$: Return of bond $i$ can take only 2 values:
  \[ R_i = + (100(1 - Y_i) - 95) = 5 - 100Y_i \]
  with the default indicator $Y_i \in \{0, 1\}$.

- $\{R_i\}_{1 \leq i \leq 50}$ is a series of iid random variables
  \[ P(R_i = 5) = 0.98 \]
  \[ P(R_i = -95) = 0.02 \]
EXAMPLE: THE PORTFOLIOS

We consider 2 different portfolios, each composed of 100 bonds and with present value 9500:

A 100 units of bond \( i \)
   (Is doesn’t matter which one, the portfolio is 100% concentrated)

B 2 units of each bond
   Portfolio B is completely diversified.
EXAMPLE: VAR OF PORTFOLIO A

\[ R = 100R_1 \text{ and therefore} \]

\[ \text{VaR}_{0.95}(R) = 100 \text{ VaR}_{R_1} = -500. \]

I.e., even with a reduction of the capital of an amount of 500 the portfolio would be accepted by a regulator who is working with \( \text{VaR}_{0.95} \).
EXAMPLE: VAR OF PORTFOLIO B

\[ R = \sum_{i=1}^{50} 2 R_i = 500 - 200 \sum_{i=1}^{50} Y_i \]

and thus

\[ \text{VaR}_\alpha(R) = -q_{1-\alpha}(F_R) = -\left[500 + 200q_{1-\alpha}(F_{\tilde{R}})\right] \quad (2) \]

with

\[ \tilde{R} = -\sum_{i=1}^{50} Y_i. \]
VAR OF PORTFOLIO B (CONT.)

\( \tilde{R} \) follows a binomial distribution with \( p = 0.2 \) and \( n = 50 \):

\[
F_{\tilde{R}}(\tilde{r}) = \sum_{i=-50}^{\tilde{r}} \binom{50}{-i} 0.02^{-i} 0.98^{50+i}.
\]

(By inspection):

\[ q_{0.05}(F_{\tilde{R}}) = -3 \]

\[ \text{VaR}_{0.95}(R) = -500 - 200 \cdot (-3) = 100 \]

Risk capital of 100 is necessary to satisfy the \( \text{VaR}_{0.95} \) requirement.
Binomialverteilung, $p=0.02$

![Graph of Binomial distribution with $p=0.02$.]

Binomialverteilung, $p=0.02$, $q=0.98$

![Graph of Binomial distribution with $p=0.02$, $q=0.98$.]
PORTFOLIOS: DISCUSSION

• BUT:
  Our economic intuition tells us that $B$ is less risky than $A$

→ There is a problem with the VaR

• Therefore introduction of coherent risk measures
AXIOMS OF COHERENT RISK MEASURES

A risk measure $\rho(L)$ of a loss $L$ is called coherent if the following axioms are satisfied:

1. Translational invariance:
   \[ \forall L \, \forall l \in \mathbb{R}: \quad \rho(L + l) = \rho(L) + l \]
   (Addition or substraction of a deterministic quantity to a risky position leads to a corresponding change of the required capital)

2. Subadditivity:
   \[ \forall L_1, L_2: \quad \rho(L_1 + L_2) \leq \rho(L_1) + \rho(L_2) \]
   (Aggregation of position doesn’t create new risks)
3. Homogeneity:
\[ \forall \text{ loss } L \forall \lambda > 0: \rho(\lambda L) = \lambda \rho(L) \]
(Is a natural consequence of A2:
\[ \rho(nL) = \rho(L + \ldots + L) \leq n\rho(L) \]
with equality if no diversification effect.)

4. Monotony:
\[ \forall L_1, L_2 \text{ with } L_1 \leq L_2: \rho(L_1) \leq \rho(L_2) \]
(Obvious from an economic point of view)

It follows from A1–A3 that A4 is equivalent to:
\[ \rho(L) \leq 0 \forall L \leq 0 \]
A2 is the most discussed axiom in the finance community. Possible reason: Some commonly used risk measure as VaR violate A2.

Arguments in favor of A2:
- Reflects idea that risk can be diminished through diversification
– If violated risk can be diminished if it is repartitioned onto subsidiaries (or at the exchange: decrease of margin requirements would be possible by opening separate accounts of each position in a portfolio)

– A2 make decentralization of risk management systems possible:
If losses of two departments are $L_1$ and $L_2$, respectively then total loss $\rho(L)$ is less than the sum of the individual risk measures.
→ Risk limit of the whole company is satisfied of the sum of the risk measure of its parts satisfy the limit.
Expected Shortfall

- Given that the less exceeds the VaR$_\alpha$, we are interested how big will the expected loss be.

- Information contained in the **Expected Shortfall** (ES).

- ES$_\alpha$ is the conditional expectation of the loss given that it exceeds the VaR$_\alpha$:

$$\text{ES}_\alpha(R) = \mathbb{E}[-R | -R \leq \text{VaR}_\alpha(R)] = -\int_{-\infty}^{q_\alpha(R)} r\phi_R(r) \, dr.$$  

The expression with the integral is valid if the pdf $\phi_R(r)$ exists.
EXPECTED SHORTFALL FOR BOND PORTFOLIOS

Ptf A  Conditional expectation of return:

\[ 0.98 \times 5 - 0.02 \times 100 = 4.9 - 2 = 2.9 \]

Thus for the concentrated portfolio:

\[ ES_{0.95} = -290 \]

Ptf B  \( VaR_{0.95} = 100 \), thus \( ES_{0.95} \leq 100 \).

Also the expected shortfall doesn’t fulfil A2!
Contrary to the VaR, the problem with the expected shortfall can easily be solved by modifying the definition of the ES in the following way:

\[
\text{GES}_\alpha = \frac{1}{1 - \alpha} \int_{\alpha}^{1} F_{-R}(u) \, du.
\]
Binomialverteilung, $p=0.02$

Binomialverteilung, $p=0.02$, $q=0.98$
Ptf A: \[
\frac{-0.02 \times 0.95}{0.05} = -38,
\]
so that for the whole portfolio
\[GES_{0.95} = 100 \times 38 = 3800.\]

Ptf B: Numerical computation shows that \(GES_{\alpha} \simeq 104\).
Thus, the GES of the diversified portfolio is much less than the GES of the concentrated portfolio, as economic intuition expects.