Contagious Defaults in Financial Networks

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Joint work with :
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A network of financial counterparties can be modeled as a *weighted directed graph*:

- **Weighted**: Each link between two institutions has a weight associated with it, representing the exposure or risk level. For instance, if Bank A lends to Bank B, the weight on the link from Bank A to Bank B could represent the maximum loss Bank A would incur if Bank B defaults.

- **Directed**: The direction of the links indicates the flow of exposures or risks. For example, a link from Bank A to Bank B implies that Bank A is exposed to Bank B, but not necessarily vice versa.

In summary, financial networks model the relationships and exposures between different financial institutions, with weights assigned to directed edges to quantify the level of risk or exposure.
A network of financial counterparties can be modeled as a weighted directed graph:

- $n$ vertices $i \in V$ represent financial market participants: banks, companies, hedge funds...

- $(i, j)$ is the exposure of $i$ to $j$, i.e. the maximum loss incurred by $i$ upon the default of $j$. Example: interbank loan from $i$ to $j$. Each institution $i$ disposes of a capital buffer $c(i)$ which absorbs market losses.
A network of financial counterparties can be modeled as a *weighted directed graph*:

- $n$ vertices $i \in V$ represent financial market participants: banks, companies, hedge funds...
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### Balance sheet

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**Table:** Stylized balance sheet of a bank.

Suppose a loss in the assets of institution $i$. $c(i)$!

Solvency condition: $c(i) > 0$. 

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**Solvency condition:** $c(i) > 0$. 
The default of a market participant $j$ affects its counterparts in the following way over a short term horizon:

Creditors lose a fraction $(1-R)$ of their exposure. Loss is first absorbed by capital: $c(i) - (c(i)(1-R)e(i,j)) +$. This leads to a writedown of $(1-R)e(i,j)$ in the balance sheet of $i$, which can lead to default if $c(i) < (1-R)e(i,j)$. Typically $R' \leq 0$ in the short term (liquidation takes time). Insolvency occurs if $\text{Loss}(i) > c(i)$. 

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Typically $R \approx 0$ in the short term (liquidation takes time). Insolvency occurs if $\text{Loss}(i) > c(i)$. 
Default cascades: a simple example
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Diagram showing nodes and edges with numerical values.
Default cascades: a simple example

A network diagram showing the connections between nodes labeled as a, b, c, d, e, f, and g. The numbers associated with the nodes represent different values, and arrows indicate the direction of influence or interaction. The specific values and connections are as follows:

- Node a connects directly to b and c with edge weights of 7 and 8, respectively.
- Node b connects to d with an edge weight of 5.
- Node c connects to e with an edge weight of 5.
- Node d connects to f with an edge weight of 10.
- Node e connects to g with an edge weight of 11.
- Node f connects to e with an edge weight of 6.
- Node g connects to e with an edge weight of 9.

The diagram illustrates how default cascades can spread through a network, with each node potentially influencing others, as indicated by the edge weights.
Default cascades: a simple example
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Contagion finishes in $T = 3$ rounds.

The set of fundamental defaults: $\{ i \mid c^0(i) = 0 \} (= \{ a \})$.

The default cluster: $\{ i \mid c^T(i) = 0 \} (= \{ a, b, c, d \})$.

Final number of defaults: $N_{def}(e, c^0)(= 4)$.
Motivation

Real financial networks are large (thousands of nodes), complex networks.

The Brazilian interbank network

Source: Cont et al. (2010)

Number of nodes $n > 2400$.

Heterogeneity in number of debtors/creditors: Out-degree/in-degree has a regularly varying distribution tail with exponent $2/3$.

Heterogeneous exposures sizes: all full of bilateral exposures are $>100$ times larger than most of the rest.
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- Number of nodes $n \approx 2400$.
- **Heterogeneity in number of debtors/creditors**: Out-degree/in-degree has a regularly varying distribution tail with exponent $\approx 2/\approx 3$.
- **Heterogeneous exposures sizes**: a handful of bilateral exposures are $> 100$ times larger than most of the rest.
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But: the statistical description is viable!
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But: the statistical description is viable!
Idea: use limit theorems to answer questions like:

- How is the magnitude of contagion related to the topology of the network and the local properties of the nodes (i.e. balance sheet composition)?
- How to stress test the resilience of a given network? (Regulatory perspective)
- How to identify the nodes posing systemic risk in terms of their local features? (Regulatory perspective)
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Financial system: network $e_n$ with the vertex set $[1, \ldots, n]$ and the corresponding sequence of capital $c_n = (c_n(i))_{i=1}^n$.

To study the asymptotics of insolvency cascades in large networks we introduce a random network ensemble of which the network $(e_n, c_n)$ may be considered as a typical sample.
Degree sequences

For any node $i$, its out-degree is given by its number of debtors $d^+_n(i) = \#\{e_n(i, j) > 0\}$ and its in-degree is given by its number of creditors $d^-_n(i) = \#\{e_n(j, i) > 0\}$. The sequences of degrees $d^+_n = (d^+_n(i))_{i=1}^n$ and $d^-_n = (d^-_n(i))_{i=1}^n$ have the property

$$\sum_{i=1}^n d^+_n(i) = \sum_{i=1}^n d^-_n(i).$$

We introduce the empirical distribution of the degrees as

$$\mu_n(j, k) := \frac{1}{n} \#\{i : d^+_n(i) = j, d^-_n(i) = k\}.$$
Random financial network

**Definition (Random network ensemble)**

Let $G_n(e_n)$ be the set of all weighted directed graphs with degree sequence $d_n^+, d_n^-$ such that, for any node $i$, the set of exposures is given by the non-zero elements of line $i$ in the exposure matrix $e_n$. On a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, we define $E_n$ as a random network uniformly distributed on $G_n(e_n)$.

We endow the nodes in $E_n$ with the capital buffers $c_n$.

We have:

$$\forall i = 1, \ldots, n,$$
$$\{E_n(i, j), \quad E_n(i, j) \neq 0\} = \{e_n(i, j), \quad e_n(i, j) \neq 0\} \quad \mathbb{P} - a.s.$$  

$$\#\{j \in v, \quad E_n(j, i) > 0\} = d_n^+(j)$$

$$\#\{j \in v, \quad E_n(i, j) \neq 0\} = d_n^-(i).$$
Asymptotic study: aim

Aim: study contagion in the random financial network $E_n$ as its size $n \to \infty$.

More precisely, we introduce the final fraction of defaults

$$\alpha_n(E_n, c_n) = \frac{N_{\text{def}}(E_n, c_n)}{n}.$$
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Question 1: Convergence in probability? $\alpha_n(E_n, c_n) \overset{p}{\to}$? and under which assumptions?
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Question 2: How does the limit depend on the network topology and the local properties of the nodes?

Question 3: Is the network resilient to the initial default of a small fraction of nodes?
Configuration Model

\{e(1, \cdot) > 0\} \quad \{e(i, \cdot) > 0\} \quad \{e(n, \cdot) > 0\}
Configuration Model

\[ c_n(1) \cdots d_n^-(1) 1 \quad 2 \cdots d_n^-(i) \cdots 2 \cdots d_n^-(n) \]

\[ 1 : c_n(1) \quad i : c_n(i) \quad \cdots \quad n : c_n(n) \]

\[ \{ e(1, \cdot) > 0 \} \quad \{ e(i, \cdot) > 0 \} \quad \{ e(n, \cdot) > 0 \} \]
Configuration Model

Configuration: a random uniform matching of the set of in-coming half edges with the set of out-going half edges.

\[
\begin{align*}
1 & : c_n(1) \\
2 & : d_n^+(1) \\
\cdots & \\
n & : c_n(n) \\
\end{align*}
\]

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This random (multi)graph has prescribed degree sequence and for each node a prescribed exposure sequence! We denote it $G^*_n(e_n)$.
Assumptions on the degree sequence

We assume that there exists a probability distribution $\mu$ on $\mathbb{N}^2$ such that:

1. The empirical proportion of nodes of degree $(j, k)$ tends to $\mu(j, k)$:

   $$\mu_n(j, k) \rightarrow \mu(j, k) \text{ as } n \rightarrow \infty;$$

2. Finite average degree property:

   $$\exists \lambda \in (0, \infty), \quad \sum_{j,k} j \mu(j, k) = \sum_{j,k} k \mu(j, k) =: \lambda;$$

3. $$\sum_{i=1}^{n} (d^+_n(i))^2 + (d^-_n(i))^2 = O(n).$$
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3. \[
   \sum_{i=1}^{n} (d_+^n(i))^2 + (d_-^n(i))^2 = O(n).
   \]

These assumptions ensure (Janson 2009) that

\[
\liminf_{n \to \infty} \mathbb{P}(G^*_n(e_n) \text{ is simple}) > 0.
\]
Default threshold

For a node $i$ and permutation $\tau \in \Sigma^e(i)$ of $i$'s counterparties, the threshold function

$$\Theta(i, e, c, \tau) := \min\{ k \geq 0, c(i) < \sum_{j=1}^{k} (1 - R)e(i, \tau(j)) \},$$

measures how many counterparty defaults $i$ can tolerate before it becomes insolvent, if its counterparties default in the order specified by $\tau$.

$$p_n(j, k, \theta) := \# \{(i, \tau) | \tau \in \Sigma^e_i, d^+_n(i) = j, d^-_n(i) = k, \Theta(i, e_n, c_n, \tau) = \theta \} / n! \mu_n(j, k) j!.$$
A link is called *contagious* if it generates a default of the starting node if the end node defaults:

\[ e_n(i, j) > c_n(i). \]

\( n\mu_n(j, k)jp_n(j, k, 1) \) is the total number of contagious links that leave a node with degree \((j, k)\).
The value \( p_n(j, k, 1) \) gives the proportion of contagious exposures belonging to nodes with degree \((j, k)\).
Assumptions on the exposure sequence

There exists a function $p : \mathbb{N}_+^3 \rightarrow [0, 1]$ such that for all $j, k, \theta \in \mathbb{N} \ (\theta \leq j)$

$$p_n(j, k, \theta) \xrightarrow{n \to \infty} p(j, k, \theta).$$

as $n \to \infty$. Then, $p(j, k, \theta)$ represents the fraction of nodes with degree $(j, k)$ and default threshold $\theta$.

This assumption is fulfilled for example in a model where exposures are exchangeable arrays.
Let us define

\[ \beta(n, \pi, \theta) := \mathbb{P}(\text{Bin}(n, \pi) \geq \theta) = \sum_{j \geq \theta} \binom{n}{j} \pi^j (1 - \pi)^{n-j}. \]
The intuition: mean field approximation

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1. Default independently with probability \( \pi \)
2. Default with probability \( I(\pi) := \sum_{j,k} \frac{k \mu(j,k)}{\lambda} \sum_{\theta} p(j, k, \theta) \beta(j, \theta, \pi) \)
3. Degree \((j, k)\) with probability \( \frac{k \mu(j,k)}{\lambda} \)
4. Randomly chosen edge
The intuition: mean field approximation

We can give for

$$I(\pi) := \sum_{j,k} \frac{k\mu(j,k)}{\lambda} \sum_{\theta=0}^{j} p(j,k,\theta)\beta(j,\pi,\theta)$$

the following interpretation: if the end node of a randomly chosen edge defaults with probability $\pi$, $I(\pi)$ is the expected fraction of counterparty defaults after one iteration of the cascade.

The companion function

$$J(\pi) := \sum_{j,k} \mu(j,k) \sum_{\theta=0}^{j} p(j,k,\theta)\beta(j,\pi,\theta),$$

gives the fraction of defaulted nodes supposing that the end-node of a randomly chosen edge defaults with probability $\pi$. 

The limit in probability for the final fraction of defaults

**Theorem**

Consider a sequence of exposure matrices and capital ratios \( \{(e_n)_{n \geq 1}, (c_n)_{n \geq 1}\} \) and the corresponding sequence of random matrices \( (E_n)_{n \geq 1} \). Let \( \pi^* \) be the smallest fixed point of function \( I \). We have
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1. If \( \pi^* = 1 \), i.e., if \( I(\pi) > \pi \) for all \( \pi \in [0, 1] \), then asymptotically all nodes default during the cascades: \( \alpha_n(E_n, c_n) = 1 - o_p(1) \).
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   nodes default during the cascades: 
   \[\alpha_n(E_n, c_n) = 1 - o_p(1).\]

2. If \(\pi^* < 1\) and furthermore \(\pi^*\) is a stable fixed point of \(I\), then the 
   asymptotic fraction of defaults is given by 
   \[\alpha_n(E_n, c_n) \xrightarrow{p} J(\pi^*) = \sum_{j,k} \mu(j, k) \sum_{\theta=0}^j p(j, k, \theta) \beta(j, \pi^*, \theta).\]
Corollary

If the resilience condition holds:

\[ \sum_{j,k} \frac{j k \mu(j, k)}{\lambda} p(j, k, 1) < 1, \]

then for every \( \varepsilon > 0 \), there exists \( N_\varepsilon \) and \( \rho_\varepsilon \) such that if the initial fraction of defaults is smaller than \( \rho_\varepsilon \), then \( \mathbb{P}(\alpha_n(E_n, c_n) \leq \varepsilon) > 1 - \varepsilon \) for all \( n \geq N_\varepsilon \).
The skeleton of contagious links

The converse also holds:

**Proposition**

If the resilience condition fails:

\[
\sum_{j,k} jk \frac{\mu(j,k)}{\lambda} p(j,k,1) > 1,
\]

then there exists a strongly connected set \(\chi_n\) of nodes representing a positive fraction of the financial system, i.e. \(|\chi_n|/n \to c > 0\) such that, with high probability, any node belonging to this set can trigger the default of all nodes in the set: for any sequence \((c_n)_{n \geq 1}\) such that \(\{i, c_n(i) = 0\} \cap \chi_n \neq \emptyset\),

\[
\liminf_n \alpha_n(E_n, c_n) \geq c > 0.
\]
Bootstrap percolation - a process of activation on a graph: each node becomes active the first time its number of active neighbors is greater than a constant threshold \( r \geq 2 \).
Bootstrap percolation - a process of activation on a graph: each node becomes active the first time its number of active neighbors is greater than a constant threshold $r \geq 2$. Initially, there is a subset $A_0 \subseteq V$ which consists of active vertices. Subsequently, in each round, if a vertex has at least $r$ of its neighbors infected, then it also becomes active and remains so forever. This is repeated until no more vertices become active. We denote the final active set by $A_f$. 
Random initial set: Chalupa, Leath and Reich (1979) (regular infinite tree); Aizenman and Lebowitz (1988), Balogh and Pete (1998), Cerf and Manzo (2002) (grids); Balogh and Bollobás (2006) (hypercube); Balogh, Peres and Pete (2006), Fontes and Schonmann (2008) (infinite trees); Balogh and Pittel (2007), Janson (2009) (random regular graphs); A. (2010) (random graphs with given vertex degrees); Janson, Łuczak, Turova and Vallier (2011) (Erdős-Rényi random graphs); A., Fountoulakis (2011) (power-law random graphs with parameter $2 < \beta < 3$): Let $a_c(n) = n^{r-3+\beta} = o(n)$. When $|A_0| \ll a_c(n)$, then typically the process does not evolve, but when $|A_0| \gg a_c(n)$, then a linear fraction of vertices is eventually infected.
In the Supervisory Capital Assessment Program, implemented by the Board of Governors of the Federal Reserve System in 2009, the top 19 banks in the US were asked to project their losses and resources under a macroeconomic shock scenario. The program determined which of the large banks needed to augment its capital base in order to withstand the projected losses.
An external shock model

$LR$ : ratio between the total interbank assets and total assets

$$c(i) = \gamma_{min} A_i \frac{1}{LR}.$$  

Under a stress test scenario, a macroeconomic shock $Z$, constant over all banks affects the banks external assets (defined as the difference between total and interbank assets). After the shock, the capital ratio becomes

$$c^{(Z)}(i) = \gamma_{min} A_i \left( 1 + \left( \frac{1}{LR} - 1 \right) (1 - Z) \right) \varepsilon(i),$$

where $\varepsilon(i)$ are independent variables with

$$\mathbb{P}(\varepsilon(i) = 1) = \varepsilon = 1 - \mathbb{P}(\varepsilon(i) = 0),$$

$\varepsilon(i) = 1$ indicating whether $i$ is in default in the stress scenario under consideration.
An infinite random scale free network

Assume $p^{(Z)}(j, k, \theta)$ does not depend on the in-degree $k$.

**Figure:** Cumulative distribution for the default threshold, Minimal capital ratio $= 8\%$, Macroeconomic shock $= 20\%$. 
Figure: Function I for increasing magnitude of the macroeconomic shock. As the common factor increases, the smallest fixed point becomes 1 and the phase transition occurs.
The finite sample

In a finite network the resilience condition becomes

$$\frac{1}{m_n} \sum_i d_n^-(i) q_n^{(Z)}(i) < 1,$$

with $m_n$ the total number of links in the network and $q_n^{(Z)}(i)$: the number of 'contagious' links of bank $i$.

**Figure:** (a)Proportion of contagious links. (b)Resilience function for varying size of macroeconomic shock in the sample and limit random network.
**Final fraction of defaults**

- **Final fraction of defaults – simulation**
- **Resilience function**

**Figure**: Final fraction of defaults.
Optimal Control of Financial Contagion
Financial networks

- \( n \) vertices \( i \in V \) represent financial market participants: banks, companies, hedge funds...
- (directed) links represent counterparty exposures: \( e(i, j) \) is the exposure of \( i \) to \( j \).
- A bank \( i \) has capital \( c(i) \) that absorbs writedowns.
Each institution $i$ has a liquidity position $l(i)$.

**Realizable liquidity**: $l(i) + s(i)$, where $s(i)$ represents the unsecured short term debt (e.g. commercial paper) of bank $i$.

The realizable liquidity and NOT the liquid assets determines whether a bank is liquid (cash-flow solvent)!
On top of write-downs related to balance-sheet exposures, creditors of defaulted banks are affected via changes in funding conditions: the unsecured short term debt available to a bank whose capital was depleted will decrease!

We capture this by setting the short term debt provided to a node $i$, $s(i, A)$ is a decreasing set function.
Therefore, consider $j$ as an additional default added to the set of defaults $\mathcal{A}$. Any other institution $i$ may default if

$$l(i) + s(i, \mathcal{A} \cup \{j\}) \leq 0.$$
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The final set of defaults at the end of the default cascade is denoted by $\mathcal{D}_f$. 
We consider that, *after the exogenous shock*, the lender of last resort injects a supplementary liquidity $\xi(i)$.

The maximum budget available the lender is denoted by $M$. 

**Distress propagation with intervention**
We consider that, *after the exogenous shock*, the lender of last resort injects a supplementary liquidity $\xi(i)$.

The maximum budget available the lender is denoted by $M$.

For a bank $i \in [n]$, the capital after intervention is given by

$$c^*(i) \leftarrow c(i) + \xi(i),$$

and its cash position by

$$l^*(i) \leftarrow l(i) + \xi(i).$$
We have the following default cascade.

### Definition (Default cascade)

Starting from the set of fundamental defaults $\mathbb{D}_0$, define $\mathbb{D}_k(\xi)$ for $k = 1, \ldots, n - 1$, as the set of institutions whose capital is insufficient to absorb losses due to defaults of institutions in $\mathbb{D}_{k-1}(\xi)$ or whose realizable liquidity becomes negative due to changes in funding conditions:

$$\mathbb{D}_k(\xi) = \{ i \mid c(i) + \xi(i) < \sum_{j \in \mathbb{D}_{k-1}(\xi)} e(i, j), \text{ or } l(i) + \xi(i) + s(i, \mathbb{D}_{k-1}(\xi)) < 0 \}.$$
The final set of defaults at the end of the default cascade with liquidity injection $\xi$ is denoted by $D_f(\xi)$. 
The final set of defaults at the end of the default cascade with liquidity injection \( \xi \) is denoted by \( D_f(\xi) \).

Under **complete information on balance sheets**, the lender of last resort faces the following deterministic optimal problem

\[
\begin{align*}
\text{Minimize}_{\xi} & \quad L(\xi) := \sum_i \xi(i) + \sum_{i \in D_f(\xi)} c(i) + \sum_{i \in D_f(\xi)} \sum_{j \in [n] \setminus D_f(\xi)} e(j, i), \\
\text{subject to} & \quad \sum_i \xi(i) \leq M.
\end{align*}
\]
The case of incomplete information

So far, we have considered the case when the controller observes the complete network. In reality, however, the controller rarely has complete information on interbank linkages.
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So far, we have considered the case when the controller observes the complete network. In reality, however, the controller rarely has complete information on interbank linkages. However, it is natural (hopefully !) to assume that, at any moment, the information set available to the regulator is richer than the information set available to the market.

Assuming zero \( t = 0 \) as a time of the initial shock (fundamental defaults), we represent by the filtration \( (\mathcal{F}_t)_{t \geq 0} \), the information flow available to the controller at time \( t \geq 0 \). Then, at each time \( t \), the controller can decide (subject to her budget contraints) to inject liquidity in any bank so as to minimize the total expected loss in the system until the cascade ends.
The case of incomplete information

- At any moment, we partition nodes that are in default into *recorded* and *economic* defaults.
- Contagion begins with a set of fundamental defaults, which are assumed to be recorded.
- Banks that would default due to exposures to the recorded defaults are called economic defaults.
- Note that there is a span between a bank’s default and the time when an exposed counterparty will record its writedown. In particular, there is a time span before a bank which is in economic default will declare bankruptcy.
- An economic default that is not bailed out by the lender of last resort becomes a recorded default and the market learns about it. An economic default that is bailed out is no longer in default.
Random financial networks

A two-tired network, with banks categorized as dealers –large and/or very inter-connected financial institutions– and non-dealers –smaller financial institutions.
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A two-tired network, with banks categorized as dealers –large and/or very inter-connected financial institutions– and non-dealers –smaller financial institutions.

The random network results form choosing uniformly at random a network in the set of networks which

- Dealer banks have \((d_1^{+}, d_1^{-})\) debtors / creditors and non-dealer banks have degree \((d_2^{+}, d_2^{-})\) debtors / creditors.
- For numerical tractability of the stochastic control problem, we assume all links carry an exposure \(w\), constant over all edges.
• We assume that among the dealers, there is an initial fraction $\alpha$ that hold illiquid assets in their portfolios, funded by short term loans. It is those banks that are susceptible to changes in funding conditions. The liquidity position of these dealer banks is assumed constant and equal to $l$.

• The other dealer banks, representing a fraction $1 - \alpha$, do not depend on short term funding, and can only default due to insolvency.

• We consider that, after a recorded default $i$, any node $j$ exposed to $i$ will record the writedown related to its exposure after a random time span, independent of everything else, $T_i(j) \sim Exp(\mu)$.
### Numerical Results

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of dealer banks</td>
<td>( n_1 = 3 )</td>
</tr>
<tr>
<td>Number of non dealer banks</td>
<td>( n_2 = 21 )</td>
</tr>
<tr>
<td>Connectivity (dealer)</td>
<td>( d_1^+ = d_1^- = 9 )</td>
</tr>
<tr>
<td>Connectivity (non dealer)</td>
<td>( d_2^+ = d_2^- = 3 )</td>
</tr>
<tr>
<td>Weight</td>
<td>( w = 1 )</td>
</tr>
<tr>
<td>Intervention budget</td>
<td>( M = 6 )</td>
</tr>
<tr>
<td>Fraction of banks depending on short-term funding</td>
<td>( \alpha = \frac{2}{3} )</td>
</tr>
<tr>
<td>Illiquidity threshold</td>
<td>( \theta^* = 1 )</td>
</tr>
<tr>
<td>Maximum distance to insolvency (dealer)</td>
<td>( \theta_1 = 4 )</td>
</tr>
<tr>
<td>Maximum distance to insolvency (non-dealer)</td>
<td>( \theta_2 = 2 )</td>
</tr>
</tbody>
</table>

**Table:** Parameters of the stylized model.
Non-anticipative control

**Figure:** Difference between value function with and without intervention. $\alpha = 0$
Intervention policy: $\alpha = 0$

**Figure:** Probability to intervene. $\alpha = 0$
Intervention policy: $\alpha = .2$

**Figure:** Probability to intervene. $\alpha = .2$
Intervention policy: $\alpha = 1/3$

**Figure:** Probability to intervene. $\alpha = 1/3$
Conclusions

- We have shown limit theorems linking the cascading behavior of a financial network to the topology of network and the local features of nodes.
- These results hold for a model flexible enough to accommodate interpretations as insolvency cascade of illiquidity cascade.
- Initial shocks seeding the insolvency cascade can be modeled as the outcome of an illiquidity cascade and price feedback effects.
- The limit theorems show convergence in probability, therefore our approach can be used for stress testing a given network. As the banks are asked to project the effect of the macroeconomic shock on their balance sheets, specific values for the quantities of interest (number of contagious links and connectivity) can be reported to the regulator and the resilience can be then assessed by our criterion.
Conclusions - cont.

- The regulator can efficiently contain insolvency contagion by focusing on fragile nodes, especially those with high connectivity and over-exposed.
- Higher capital requirements could be imposed on them to reduce their number of contagious links and insure that the danger of phase transitions as described above is avoided.
The intervention policy strongly depends on the proportion of banks that use short-term funding.

When a sufficiently large proportion of banks depend on unstable short term funding, we find that it is optimal for a lender of last resort to inject liquidity even when banks that are not in immediate danger of insolvency.

The transition between optimal intervention policies occurs very sharply with a small increase in the fraction of banks that are prone to short term creditor runs.
THANK YOU!